Assignment 6

1. On $(-1,1]$ define $\rho(x,y) = 1 - |1 - |x-y||$. Show that ρ is a metric and $((-1,1],\rho)$ is compact.

2. For $f \in C_0(\mathbb{R})$, let $\tilde{f}(x) = f\left(\frac{x}{1-|x|}\right)$. Show that $\{\tilde{f} : f \in C_0(\mathbb{R})\}$ is a subspace of $C(-1,1]$, the space of continuous functions on the metric space $((-1,1],\rho)$. Which $g \in C(-1,1]$ arise as $\tilde{f}$ for some $f \in C_0(\mathbb{R})$?

3. Let $f \in C(\mathbb{R})$. Show that $f(K)$ is a compact subset of $\mathbb{C}$ for any compact $K \subset \mathbb{R}$.

4. Let $\mathbb{R}^\infty = \{x = (x_n), x_n \in \mathbb{R} : \|x\|^2 = \sum_1^\infty |x_n|^2 < \infty\}$. Let $S = \{x \in \mathbb{R}^\infty : \|x\| \leqslant 1\}$. Show that S is bounded but not totally bounded in the metric space $(\mathbb{R}^\infty, d)$, where $d(x,y) = \|x-y\|$.